On the Relationship between the Hausdorff Distance and Matrix Distances of Ellipsoids

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Dedicated to Sasha Ostrowski on his 90th birthday.
His work and life have always been an inspiration.

Submitted by Walter Gautschi

ABSTRACT

The space of ellipsoids may be metrized by the Hausdorff distance or by the sum of the distance between their centers and a distance between matrices. Various inequalities between metrics are established. It follows that the square root of positive semidefinite symmetric matrices satisfies a Lipschitz condition, with a constant which depends only on the dimension of the space.

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1. DISTANCE BETWEEN ELLIPSOIDS AS SETS

Ellipsoids in R^n may be viewed as elements of the set of subsets of R^n, subsets which could be restricted to be compact, convex, and centrally symmetric. The set of subsets of R^n is usually metrized by the Hausdorff metric [2]:

$$\delta(E,F)=\max\left\{\sup_{y\in F}\inf_{x\in E}\|x-y\|,\ \sup_{x\in E}\inf_{y\in F}\|x-y\|\right\}$$

$$=\inf\{\delta\geqslant 0: E+\delta S\supset F,\ F+\delta S\supset E\},$$

where E and F are subsets of R^n, $\| \ \|$ represents the Euclidean norm, and $S=\{x\in R^n:\|x\|\leqslant 1\}$ is the unit ball.

If E and F are convex, then

$$\delta(E,F)=\sup\{|h(x,E)-h(x,F)|:\|x\|=1\},$$

where $h(x,E)=\sup\{(x,y):y\in E\}$ is the support function of E (see Bonnensen and Fenchel [1]) and $(\cdot,\cdot)$ denotes the scalar product.

If E and F are convex and contain the origin in their interiors, then

$$\delta(E,F)=\sup\{|g(x,E^d)-g(x,F^d)|:\|x\|=1\},$$

where $E^d=\{x\in R^n:(x,y)\leqslant 1\ \forall y\in E\}$ is the dual of E, and

$$g(x,E^d)=\inf\{\mu\geqslant 0: x\in\mu E^d\}$$

is the distance function, or gauge, of E^d; this follows because $h(x,E)=g(x,E^d)$.

If E and F are convex, full-dimensional, and centrally symmetric with respect to the origin, then E^d and F^d inherit the same properties, and $g(x,E^d)$ and $g(x,F^d)$ define norms on R^n. Thus $\delta(E,F)$ may be viewed as a distance between norms on R^n.

The Hausdorff distance is invariant under congruent, but not affine, transformations, and reduced by projection. It will be assumed throughout that the space of ellipsoids contains the degenerate ellipsoids. The space of ellipsoids is not closed under addition.

The following lemma indicates that it will be sufficient to study ellipsoids centered at the origin.

LEMMA 1. *Let E and F be two subsets of R^n, compact, convex, and symmetric with respect to the origin; let $E^1 = e + E$ and $F^1 = f + F$. Then*

$$\delta(E^1, F^1) \leqslant \delta(E, F) + \|e - f\| \leqslant 2\delta(E^1, F^1),$$

$$\delta(E, F) \leqslant \delta(E^1, F^1), \qquad \|e - f\| \leqslant \delta(E^1, F^1).$$

Proof.

$$\begin{aligned}\delta(E^1, F^1) &= \sup\{|h(x, E^1) - h(x, F^1)| : \|x\| = 1\}\\ &= \sup\{|h(x, E) - h(x, F) + (e - f, x)| : \|x\| = 1\}\\ &\leqslant \sup\{|h(x, E) - h(x, F)| : \|x\| = 1\} + \sup\{|(e - f, x)| : \|x\| = 1\}\\ &= \delta(E, F) + \|e - f\|.\end{aligned}$$

Conversely

$$-\delta(E^1, F^1) \leqslant h(x, E) - h(x, F) + (e - f, x) \leqslant \delta(E^1, F^1) \qquad \forall \|x\| = 1;$$

now $h(x, E) = h(-x, E)$ and $h(x, F) = h(-x, F)$, as E and F are symmetric with respect to the origin, and thus

$$-\delta(E^1, F^1) \leqslant h(x, E) - h(x, F) - (e - f, x) \leqslant \delta(E^1, F^1) \qquad \forall \|x\| = 1.$$

Adding and subtracting, one gets

$$\begin{aligned}-\delta(E^1, F^1) &\leqslant h(x, E) - h(x, F) \leqslant \delta(E^1, F^1),\\ -\delta(E^1, F^1) &\leqslant (e - f, x) \qquad\qquad\;\; \leqslant \delta(E^1, F^1) \qquad \forall \|x\| = 1,\end{aligned}$$

and hence $\delta(E, F) \leqslant \delta(E^1, F^1)$ and $\|e - f\| \leqslant \delta(E^1, F^1)$. ∎

2. ELLIPSOIDS AS VECTORS AND MATRICES

An ellipsoid may also be represented by a vector (its center) and a matrix (its size, shape, and position):

$$E = e + \bar{A}S = \{x \in R^n : x = e + \bar{A}t, \|t\| \leqslant 1\};$$

note that $h(x, E)=(e, x)+\|\bar{A}^T x\|$. If $\bar{A}$ is nonsingular, then

$$E=\{x \in R^n:(x-e)^T(\bar{A}\bar{A}^T)^{-1}(x-e) \leqslant 1\}.$$

To any ellipsoid is associated an equivalence class of matrices; in fact $E=e+\bar{A}S=e+\bar{\bar{A}}S$ if and only if $\bar{\bar{A}}=\bar{A}O$ where O is an orthogonal matrix, or equivalently if $\bar{A}\bar{A}^T=\bar{\bar{A}}\bar{\bar{A}}^T$. Define now $H=\bar{A}\bar{A}^T$, and $A=H^{1/2}$; then in the remainder of this paper an ellipsoid will be defined by

$$E=e+AS=e+H^{1/2}S,$$

where A and H are positive semidefinite real symmetric matrices. Using any of these two definitions, there exists a one-to-one correspondence between ellipsoids and points (e, A) in $R^n \times p(R^n)$ [respectively $(e, H) \in R^n \times p(R^n)$], where $p(R^n)$ is the set of $n \times n$ positive semidefinite symmetric matrices.

One could have tried to associate to an ellipsoid a lower triangular matrix $L(H=LL^T)$; L is unique if H is nonsingular, but not necessarily so if H is singular. This is the key reason why the results of this paper will not extend to the case of Cholesky factors.

If A is nonsingular, then

$$\begin{aligned} E &= \{x \in R^n:(x-e)^T A^{-2}(x-e) \leqslant 1\} \\ &= \{x \in R^n:(x-e)^T H^{-1}(x-e) \leqslant 1\}. \end{aligned}$$

We may now define two matric distances on the space of ellipsoids.

Let $E=e+AS=e+H^{1/2}S$ and $F=f+BS=f+K^{1/2}S$ be two ellipsoids in R^n, where A, B, H, and K are positive semidefinite symmetric matrices; then define

$$\begin{aligned} & d(E, F)=\|e-f\|+\|A-B\|, \\ & \Delta(E, F)=\|e-f\|+\|H-K\|^{1/2}=\|e-f\|+\|A^2-B^2\|^{1/2}, \end{aligned}$$

where $\| \ \|$, for matrices, is the spectral norm.

It is clear that d and Δ satisfy the axioms for a metric (or distance).

Various inequalities between d, Δ, and δ will be proven in the next section; the relationship between d and δ is the closest one, as d and δ are related by inequalities involving constants depending only upon the dimension of the space.

The inequalities imply that the three metrics define the same topology on the space of ellipsoids, but, more strongly, that rates of convergence can be related.

The inequalities between d and δ imply that the rates of convergence of a sequence of ellipsoids may be studied within a space of sets, or a space of matrices, and that the two rates are identical.

3. INEQUALITIES BETWEEN DISTANCES

If E and F are ellipsoids centered at the origin, and $E^1 = e + E$, $F^1 = f + F$, then

$$d(E^1, F^1) = \|e - f\| + d(E, F),$$
$$\Delta(E^1, F^1) = \|e - f\| + \Delta(E, F);$$

and Lemma 1 indicates that it is enough to study ellipsoids centered at the origin. In that case,

$$d(E, F) = \|A - B\|,$$
$$\Delta(E, F) = \|H - K\|^{1/2} = \|A^2 - B^2\|^{1/2},$$
$$\delta(E, F) = \sup\{|\|Ax\| - \|Bx\|| : \|x\| = 1\}.$$

THEOREM 2. *Let $E = AS$ and $F = BS$ be two ellipsoids in R^n, centered at the origin, where A and B are $n \times n$ positive semidefinite symmetric matrices. Then*

$$k_n^{-1}\|A - B\| \leqslant \sup\{|\|Ax\| - \|Bx\|| : \|x\| = 1\} \leqslant \|A - B\|,$$

or

$$\delta(E, F) \leqslant d(E, F) \leqslant k_n \delta(E, F),$$

where $k_n = 2\sqrt{2}\, n(n+2)$.

Proof. For the first part, one has

$$|\|Ax\| - \|Bx\|| \leqslant \|Ax - Bx\| = \|(A - B)x\| \leqslant \|A - B\|\|x\|,$$

and $\sup\{|\|Ax\| - \|Bx\|| : \|x\| = 1\} \leqslant \|A - B\|$.

For the second part, let $\delta = \sup\{|\|Ax\| - \|Bx\||: \|x\| = 1\}$, and $\alpha_n \leqslant \alpha_{n-1} \leqslant \cdots \leqslant \alpha_1, \beta_n \leqslant \beta_{n-1} \leqslant \cdots \leqslant \beta_1$ be the ordered eigenvalues of A and B (clearly all real and nonnegative numbers).

The maximum characterization for the eigenvalues of Hermitian matrices gives

$$\alpha_k^2 = \max_{S_k} \min_{x \in S_k} x^T A^2 x,$$

where S_k represents the intersection of any k-dimensional subspace with the unit spherical surface; assume that S_k^* gives the maximum. Now define x_k^* by

$$x_k^{*T} B^2 x_k^* = \min_{x \in S_k^*} x^T B^2 x.$$

Thus

$$\beta_k^2 = \max_{S_k} \min_{x \in S_k} x^T B^2 x \geqslant \min_{x \in S_k^*} x^T B^2 x = \|Bx_k^*\|^2$$

and

$$\alpha_k^2 = \min_{x \in S_k^*} x^T A^2 x \leqslant \|Ax_k^*\|^2;$$

it follows that

$$\alpha_k - \beta_k \leqslant \|Ax_k^*\| - \|Bx_k^*\| \leqslant \delta.$$

Reversing the argument, $\beta_k - \alpha_k \leqslant \delta$, and $|\alpha_k - \beta_k| \leqslant \delta \ \forall k = 1, \ldots, n$.

The content of the theorem is unchanged if A is replaced by $O^T A O$ and B by $O^T B O$, where O is any orthogonal matrix; hence we may assume that A is diagonal, and that

$$\alpha_i = a_{ii} \qquad \forall i = 1, \ldots, n.$$

Denote by $B_k = Be_k$ the kth column of B, where e_k is the kth column of the identity matrix; then

$$|\alpha_k - \|B_k\|| = |\|Ae_k\| - \|Be_k\|| \leqslant \delta \qquad \forall k = 1, \ldots, n.$$

Hence

$$|\beta_k - \|B_k\|| \leqslant |\beta_k - \alpha_k| + |\alpha_k - \|B_k\|| \leqslant 2\delta \qquad \forall k = 1,\dots,n.$$

Now

$$\|B_k\|^2 = \left(\sum_{i=1}^{n} b_{ik}^2\right) \geqslant b_{kk}^2;$$

thus

$$0 \leqslant \|B_k\| - b_{kk},$$

$$0 \leqslant \sum_{k=1}^{n} (\|B_k\| - b_{kk}) = \sum_{k=1}^{n} (\|B_k\| - \beta_k)$$

$$\leqslant \sum_{k=1}^{n} |\|B_k\| - \beta_k| \leqslant 2n\delta,$$

implying that

$$0 \leqslant \|B_k\| - b_{kk} \leqslant 2n\delta \qquad \forall k = 1,\dots,n.$$

This leads to

$$|\alpha_k - b_{kk}| \leqslant |\alpha_k - \beta_k| + |\beta_k - \|B_k\|| + |\|B_k\| - b_{kk}| \leqslant (2n+3)\delta.$$

Let $D = \mathrm{diag}(b_{kk})$, and x be any vector of unit length; then

$$|\|Bx\| - \|Dx\|| \leqslant |\|Bx\| - \|Ax\|| + |\|Ax\| - \|Dx\||$$
$$\leqslant \delta + \|A - D\| \leqslant \delta + (2n+3)\delta = (2n+4)\delta,$$

as $\|A - D\| = \max_{k=1,\dots,n} |\alpha_k - b_{kk}| \leqslant (2n+3)\delta$.

It remains to show that the off-diagonal elements of B are bounded by a multiple of δ. If $b_{ii} = 0$ or $b_{kk} = 0$, then $b_{ik} = 0$ $(i \neq k)$, as $b_{ik}^2 \leqslant b_{ii} b_{kk}$ by the positive semidefiniteness of B. So assume that $b_{ii} > 0$, $b_{kk} > 0$, and let $a = b_{ii}$, $b = b_{kk}$, $c = |b_{ik}|$, and $\sigma = +1$ (-1) if b_{ik} is positive (negative).

Choose

$$z = \frac{1}{\sqrt{a^2+b^2}}(be_i + \sigma a e_k);$$

then

$$\begin{aligned}
\|Bz\| &= \frac{1}{\sqrt{a^2+b^2}}\|bB_i + \sigma a B_k\| \\
&= \frac{1}{\sqrt{a^2+b^2}}\left\|(ab+ac)e_i + \sigma(ab+bc)e_k + \sum_{j\neq i,k}(bb_{ji} + \sigma ab_{jk})e_j\right\| \\
&\geqslant \frac{1}{\sqrt{a^2+b^2}}\|(ab+ac)e_i + \sigma(ab+bc)e_k\| \\
&= \frac{1}{\sqrt{a^2+b^2}}\left[(ab+ac)^2 + (ab+bc)^2\right]^{1/2} \\
&= \left(c^2 + 2\frac{abc(a+b)}{a^2+b^2} + 2\frac{a^2b^2}{a^2+b^2}\right)^{1/2} \\
&= \left(\frac{\sqrt{2}\,ab}{\sqrt{a^2+b^2}} + d\right),
\end{aligned}$$

where this last equation defines d ($d > 0$).

Now, as $\|Dz\| = \sqrt{2}\,ab/\sqrt{a^2+b^2}$, it follows that $d \leqslant \|Bz\| - \|Dz\| \leqslant (2n+4)\delta$.

The value of d is given by the positive root of

$$d^2 + \frac{2\sqrt{2}\,ab}{\sqrt{a^2+b^2}}d = c^2 + \frac{2abc(a+b)}{a^2+b^2};$$

the left-hand side increases with d ($d \geqslant 0$) and is less than the right-hand side for $d = 0$ and $d = c/\sqrt{2}$, implying that the value of d is greater than $c/\sqrt{2}$, and

$$c < d\sqrt{2} \leqslant 2\sqrt{2}\,(n+2)\delta.$$

Thus $|b_{ik}| \leqslant 2\sqrt{2}\,(n+2)\delta \ \forall i, k,\ i \neq k$, and

$$\begin{aligned}\|A-B\|^2 &\leqslant \operatorname{Tr}(A-B)^2\\ &= \sum_k (a_{kk}-b_{kk})^2 + \sum\sum_{i\neq k} b_{ik}^2\\ &\leqslant n(2n+3)^2\delta^2 + n(n-1)8(n+2)^2\delta^2\\ &\leqslant 8n^2(n+2)^2\delta^2;\end{aligned}$$

hence $\|A-B\| \leqslant 2\sqrt{2}\,n(n+2)\delta$. ■

The next result, which compares the distances δ and Δ, uses an operator-theory proof, and hence carries to infinite-dimensional Hilbert spaces.

THEOREM 3. *Let $E = H^{1/2}S$ and $F = K^{1/2}S$ be two ellipsoids in R^n, centered at the origin, where H and K are positive semidefinite symmetric matrices. Then*

$$\delta(E,F) \leqslant \|H-K\|^{1/2} \leqslant \left[\delta^2(E,F) + \delta(E,F)\max(D(E),D(F))\right]^{1/2},$$

where $\delta(E,F) = \sup\{|(x^THx)^{1/2} - (x^TKx)^{1/2}| : \|x\| = 1\}$, $\Delta(E,F) = \|H - K\|^{1/2}$, *and* $D(E) = 2\|H\|^{1/2}$ *is the diameter of* E; *this may also be written as*

$$\frac{\|H-K\|}{[\|H-K\| + \max(\|H\|,\|K\|)]^{1/2} + [\max(\|H\|,\|K\|)]^{1/2}} \leqslant \delta(E,F) \leqslant \|H-K\|^{1/2}$$

Proof. Let $\delta = \delta(E,F)$; thus

$$(x^THx)^{1/2} - (x^TKx)^{1/2} \leqslant \delta\|x\| \qquad \forall x;$$

hence

$$\begin{aligned} x^THx &\leqslant \delta^2\|x\|^2 + 2\delta\|x\|(x^TKx)^{1/2} + x^TKx \qquad \forall x,\\ &\leqslant \delta^2\|x\|^2 + x^TKx + \varepsilon^{-1}\delta^2\|x\|^2 + \varepsilon(x^TKx) \qquad \forall x \quad \forall \varepsilon > 0\\ &= \delta^2(1+\varepsilon^{-1})\|x\|^2 + (1+\varepsilon)(x^TKx) \qquad \forall x, \quad \forall \varepsilon > 0.\end{aligned}$$

We have

$$x^T(H-K)x \leqslant x^T\left[\delta^2(1+\varepsilon^{-1})I+\varepsilon K\right]x \qquad \forall x, \quad \forall \varepsilon>0,$$

and similarly, reversing the argument,

$$x^T(H-K)x \geqslant -x^T\left[\delta^2(1+\eta^{-1})I+\eta H\right]x \qquad \forall x, \quad \forall \eta>0.$$

These two equations imply that

$$\|H-K\| \leqslant \max\{\varepsilon\|K\|+\delta^2(1+\varepsilon^{-1}), \eta\|H\|+\delta^2(1+\eta^{-1})\} \qquad \forall \varepsilon>0, \quad \forall \eta>0;$$

taking $\varepsilon=\delta/\|K\|^{1/2}$ and $\eta=\delta/\|H\|^{1/2}$, one gets

$$\|H-K\| \leqslant \delta^2+2\delta\max(\|H\|^{1/2},\|K\|^{1/2}).$$

For the second part, let $\Delta^2=\|H-K\|$, thus

$$\left|x^T(H-K)x\right| \leqslant \Delta^2\|x\|^2 \qquad \forall x;$$

using the inequality

$$|a-b| \leqslant \sqrt{|a^2-b^2|} \qquad (a,b\geqslant 0)$$

one gets

$$\left|(x^THx)^{1/2}-(x^TKx)^{1/2}\right| \leqslant \Delta\|x\| \qquad \forall x,$$

and

$$\begin{aligned}\delta(E,F)&=\sup\{\left|(x^THx)^{1/2}-(x^TKx)^{1/2}\right| : \|x\|=1\}\\&\leqslant \Delta=\|H-K\|^{1/2}.\end{aligned}$$

■

Theorems 2 and 3 can be combined to give a relationship between the distances d and Δ, which is a statement about square roots of matrices.

THEOREM 4. *Let H and K be two $n \times n$ positive semidefinite matrices, and $A = H^{1/2}$, $B = K^{1/2}$. Then*

$$l_n^{-1}\|A - B\| \leqslant \|H - K\|^{1/2} \leqslant \left[2\|A - B\|\max(\|A\|,\|B\|) + \|A - B\|^2\right]^{1/2},$$

or

$$\frac{\|H - K\|}{\left[\|H - K\| + \max(\|H\|,\|K\|)\right]^{1/2} + \left[\max(\|H\|,\|K\|)\right]^{1/2}} \leqslant \|A - B\| \leqslant l_n\|H - K\|^{1/2},$$

where $l_n = k_n = 2\sqrt{2}\,n(n+2)$.

This theorem means that the square root satisfies a Lipschitz condition on the cone of positive semidefinite matrices:

$$\|H^{1/2} - K^{1/2}\| \leqslant l_n\|H - K\|^{1/2} \qquad \forall H, K \in p(R^n),$$

where the Lipschitz constant depends only upon the dimension of R^n; l_n is bounded by a polynomial of degree 1 in the dimension of $p(R^n)$.

It is now a simple matter to extend Theorems 2, 3, and 4 to the case of ellipsoids not necessarily centered at the origin.

THEOREM 5. *Let $E = e + AS = e + H^{1/2}$ and $F = f + BS = f + K^{1/2}S$ be two ellipsoids in R^n, and A, B, H, and K be $n \times n$ positive semidefinite symmetric matrices. Denote $\delta = \delta(E, F)$, $d = d(E, F)$, $\Delta = \Delta(E, F)$, and $M = \max(\|A\|,\|B\|) = \max(\|H\|^{1/2},\|K\|^{1/2}) = \frac{1}{2}\max(D(E), D(F))$. Then the following inequalities are satisfied:*

$$(k_n + 1)^{-1}d \leqslant \delta \leqslant d \leqslant (k_n + 1)\delta,$$

$$l_n^{-1}d \leqslant \Delta \leqslant (d^2 + 2dM)^{1/2},$$

$$\frac{\Delta^2}{\sqrt{\Delta^2 + M^2} + M} \leqslant d \leqslant l_n\Delta,$$

$$\delta \leqslant \Delta \leqslant \delta + (\delta^2 + 2\delta M)^{1/2},$$

$$\frac{\Delta^2}{2(M + \Delta)} \leqslant \delta \leqslant \Delta$$

with $k_n = l_n = 2\sqrt{2}\,n(n+2)$.

Proof. Let $\varepsilon = \|e - f\|$, and δ_0, d_0, and Δ_0 be the distances between $E - e$ and $F - f$.

One has $d = d_0 + \varepsilon$, $\Delta = \Delta_0 + \varepsilon$ and, by Lemma 1, $\delta \leqslant \delta_0 + \varepsilon$, $\varepsilon \leqslant \delta$, and $\delta_0 \leqslant \delta$. Hence a slight difference appears in the proofs for the various cases.

For instance, Theorem 4 implies

$$\Delta_0 \leqslant \left(d_0^2 + 2d_0 M\right)^{1/2}.$$

Hence $\Delta = \Delta_0 + \varepsilon \leqslant (d_0^2 + 2d_0M)^{1/2} + \varepsilon$; the maximum of the right-hand side (subject to $d_0 \geqslant 0$, $\varepsilon \geqslant 0$, and $\varepsilon + d_0 = d$) is attained for $\varepsilon = 0$ and $d_0 = d$, and thus

$$\Delta \leqslant \left(d^2 + 2dM\right)^{1/2},$$

or

$$d \geqslant \frac{\Delta^2}{\sqrt{\Delta^2 + M^2} + M}.$$

The equivalent result from Theorem 3 implies

$$\Delta_0 \leqslant \left(\delta_0^2 + 2\delta_0 M\right)^{1/2};$$

hence

$$\Delta = \Delta_0 + \varepsilon \leqslant \left(\delta_0^2 + 2\delta_0 M\right)^{1/2} + \varepsilon,$$

and the maximum of the right-hand side subject to $\varepsilon \leqslant \delta$ and $\delta_0 \leqslant \delta$ is clearly attained for $\varepsilon = \delta$ and $\delta_0 = \delta$, and thus

$$\Delta \leqslant \left(\delta^2 + 2\delta M\right)^{1/2} + \delta,$$

or

$$\delta \geqslant \frac{\Delta^2}{2(M + \Delta)}.$$

The other cases follow similarly. ∎

4. CONCLUSION

Three metrics on the space of ellipsoids have been shown to be linked by various inequalities, and hence the induced topologies are identical. Not only is the notion of convergence unique, but rates of convergence can be related. Similar results clearly hold if the Euclidean norm is replaced by any of the L_p norms.

If k_n and l_n were defined to be the smallest constants satisfying Theorems 2 and 4 (with $l_n \leqslant k_n$), it would be quite interesting to know whether or not they must depend on n, the dimension.

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